Quantum entanglement measurement based on belief entropy☆

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Abstract

Partial entropy entanglement is a very popular method to measure the entanglement of quantum systems, which is based on the classic von Neumann entropy. However, because of the problem of classical von Neumann entropy in measuring the uncertainty of quantum systems, the partial entropy entanglement is not efficient enough to measure the entanglement of quantum systems. The new entropy measure of quantum uncertainty is a model for measuring the uncertainty of a quantum system based on the classic von Neumann entropy and belief entropy, which has higher performance than the classic von Neumann entropy in measuring the uncertainty entropy of a quantum system. Based on the new entropy measure of quantum uncertainty and the classic partial entropy entanglement, this paper proposes a new model to measure the quantum entanglement measurement, named quantum entanglement measurement based on belief entropy. When the new entropy measure of quantum uncertainty degenerates to classical von Neumann entropy, the quantum entanglement measurement based on belief entropy will degenerate to classical partial entropy entanglement. Numerical examples are used to prove that quantum entanglement measurement based on belief entropy is more efficient and reliable in measuring the entanglement of quantum systems than the classic partial entropy entanglement. The experimental results show that the quantum entanglement measurement based on belief entropy can measure the uncertainty of quantum systems more efficiently and reliably than the classical classic partial entropy entanglement.

Keywords: Quantum system, Belief entropy, Von Neumann entropy, Uncertainty, Entanglement, Partial entropy entanglement

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1. Introduction

There is a lot of unknown information in the unknown world [1, 2, 3, 4]. In order to process unknown information, it is necessary to express the unknown first, so many scholars have proposed many mathematical methods and theories for expressing unknown information [5, 6, 7, 8]. For example, Song et al. [9] proposed a novel combination method for temporal evidence. Pan et al. [10] proposed a new Pythagorean fuzzy sets and its similarity measure. After the unknown information is represented reasonably, many experts propose models and methods to deal with the unknown information [11, 12, 13]. For example, Deng and Jiang [14] applied the maximum uncertainty allocation to improve Dempster–Shafer belief structure. Garg et al. [15, 16] used entropy theory to improve measurement methods under Pythagorean fuzzy environment. Prajapati and Saha [17] applied the entropy theory to predict next word in the text with the aid of language model. Abellan [18] analyzed the properties of belief entropy in evidential environment. Zhu [19] proposed the maximum value dimension and power law of belief distribution of the maximum belief entropy. Kang and Deng [20] proposed the maximum belief entropy. Gao and Deng [21] proposed the Pseudo-Pascal Triangle form for the maximum belief entropy.

Quantum theory is the most popular theory in recent years [22, 23, 24, 25], and it has been studied by many experts [26, 27, 28]. A quantum system is a system based on quantum theory [29, 30, 31], which is different from traditional systems and has many characteristics [32, 33, 34], such as quantum uncertainty, superposition of quantum states, and entanglement of quantum states [35, 36, 37]. So how to evaluate the entanglement of a quantum system has always been an important issue recognized by academia [38, 39, 40]. In response to this problem, many experts put forward their own methods and models to measure the entanglement of quantum systems [41, 42, 43]. For example Xavier and Rajabpour [44] proposed the entanglement and boundary entropy in quantum spin chains. Karabali [45] proposed a new entanglement entropy for integer quantum Hall effect under two and higher dimensions. Alimuddin et al. [46] studied the independence of entropy and work of equal-energetic finite quantum systems. Moitra and Sensarma [47] proposed the entanglement entropy of fermions from Wigner functions. Grimmett et al. [48] proposed the bounded entanglement entropy under quantum ising model. Verga and Elas [49] proposed the thermal state entanglement entropy under quantum graph. Hirano [50] proposed a new entanglement entropy for all orders in 1/N expansion. In these methods and models, partial entropy entanglement is one of the most popular models [51]. Many experts have conducted research on partial entropy entanglement [52]. For example, Han and Kye [53] proposed the convex cones under classifications of partial entanglement in the three qubit system. Huber et al. [54] proposed high-dimensional entanglement in states under positive partial transposition. Dwivedi et al. [55] proposed multi-boundary entanglement in Chern-Simons theory based on finite gauge groups. Shapourian and Ryu [56] proposed finite-temperature entanglement negativity of free fermions. Bauml et al. [57] studied the fundamental limits on the capacities of bipartite quantum

interactions.

Recently, Xue and Deng [58] proposed a new entropy measure of quantum system uncertainty, which is based on von Neumann entropy and belief entropy.Compared with the classical von Neumann entropy, the new entropy measure of quantum system uncertainty can better measure the uncertainty of quantum systems. Because the new entropy measure of quantum system uncertainty has advantages in measuring the uncertainty of quantum systems that the classical von Neumann entropy does not have, the new entropy measure of quantum system uncertainty has a good research prospect

This paper proposed the quantum entanglement measurement based on belief entropy, which is based on the new entropy measure of quantum system uncertainty and partial entropy entanglement. The proposed model requires the entanglement degree of the quantum system to be obtained from the uncertainty of the quantum system and its subsystems. In other words, the uncertainty of all subsystems of the quantum system minus the uncertainty of the quantum system divided by 2 is the entanglement of the quantum system. The advantage of the proposed model over partial entropy entanglement is that the proposed model is based on new entropy measure of quantum uncertainty. The new entropy measure of quantum uncertainty is a better model than the classical von Neumann entropy in measuring the uncertainty of quantum systems. When the new entropy measure of quantum uncertainty degenerates to classical von Neumann entropy, the quantum entanglement measurement based on belief entropy will degenerate to classical partial entropy entanglement.

The remaining of this paper is structured as follows. Section 2 introduces the preliminary. Section 3 presents the quantum entanglement measurement based on belief entropy. Section 4 illustrates the flexibility and accuracy of the quantum entanglement measurement based on belief entropy. Section 5 summarizes the whole paper.

2. Preliminaries

Quantum systems are full of uncertainties [59, 60, 61]. In order to deal with these uncertainties, many models and theories have been proposed [62, 63, 64].

2.1. Belief entropy

Given m is a mass function on frame of discernment $X = \{x_1, \ldots, x_n\}$, the definition of belief entropy is as follows:

Definition 2.1. *(Belief entropy) [65]*

$$E_d = -\sum_{A \subseteq X} m(A) \log \frac{m(A)}{2^{|A|} - 1} \tag{1}$$

Belief entropy is the generalization of Shannon entropy [66].

2.2. Von Neumann Entropy

Quantum theory is a famous field [67, 68, 69], which has attracted the attention of many scholars [70, 71, 72, 73]. Von Neumann entropy, as a classic method to evaluate the uncertainty of quantum systems, has attracted the attention and research of many scholars [74]. Given a density matrix, ρ, the definition of von Neumann entropy of ρ is as follows:

Definition 2.2. *(Von Neumann Entropy) [75]*

$$S(\rho) = -Tr[\rho \ln \rho] \tag{2}$$

The von Neumann entropy also can be defined as follows:

$$S(\rho) = -\sum_i \lambda_i \ln \lambda_i \tag{3}$$

where, $0 * \ln 0 = 0$. $\lambda_i, i \in \{1, 2, \ldots, m\}$ *is the eigenvalue of* ρ.

2.3. A new entropy measure of quantum uncertainty

Definition 2.3. *(A new entropy measure of quantum system uncertainty) [58] Given a density matrix,* ρ*, of a quantum system, the definition of the new entropy measure of* ρ *is as follow:*

$$D(\rho) = -\sum_i \lambda_i \ln \frac{\lambda_i}{2^{|\lambda_i|} - 1} \tag{4}$$

where, λ_i *is the eigenvalue of* ρ. $|\lambda_i|$ *is the cardinality of* ρ*, which means that* $|\lambda_i|$ *represents how many orthogonal ground states the eigenvector corresponding to* λ_i *is composed of.*

Example 2.1. *Suppose the quantum system at this time is a pure state, as shown below:*

$$|\psi >= \frac{6e^{i\alpha}}{10}|10> + \frac{8e^{i\alpha}}{10}|01>$$

where,

$$|10 >= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, |01 >= \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

The density matrix obtained from the above is as follows:

$$\rho = |\psi >< \psi| = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{36}{100} & \frac{48}{100} & 0 \\ 0 & \frac{48}{100} & \frac{64}{100} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The eigenvalues and the corresponding eigenvectors of ρ *are in following Table 1:*

Table 1: The eigenvalues and the corresponding eigenvectors of ρ

$Eigenvalues$	1	0	0	0
$Eigenvectors$	$\begin{pmatrix} 0 \\ 0.6 \\ 0.8 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 0.8 \\ -0.6 \\ 0 \end{pmatrix}$

Relying on the equation of Eq.(4), the equations are obtained:

$$D(\rho) = -\sum_i \lambda_i \ln \frac{\lambda_i}{2^{|\lambda_i|} - 1} = ln3$$

2.4. Partial entropy entanglement

Definition 2.4. *(Partial entropy entanglement) [51] Given a density matrix, ρ, of a quantum system, the definition of the partial entropy entanglement of ρ is as follow:*

$$E_l(\rho_{AB}) = \frac{1}{2}\{S(\rho_A) + S(\rho_B) - S(\rho_{AB})\} \tag{5}$$

where, $S(\rho_A)$, $S(\rho_B)$, $S(\rho_{AB})$ are the von Neumann entropy of subsystem A, subsystem B and two-body quantum system AB, respectively.

3. The proposed method

The world is uncertain [76, 77, 78], which causes a lot of issues [79, 80, 81]. The quantum system is flexibel, which means that more effective model should be proposed [82, 83, 84].

Definition 3.1. *(Quantum entanglement measurement based on belief entropy) Given a density matrix, ρ, of a quantum system, the definition of the quantum entanglement measurement based on belief entropy of ρ is as follow:*

$$E_D(\rho_{AB}) = \frac{1}{2}\{D(\rho_A) + D(\rho_B) - D(\rho_{AB})\} \tag{6}$$

where, $S(\rho_A)$, $S(\rho_B)$, $S(\rho_{AB})$ are the new entropy measure of quantum uncertainty of subsystem A, subsystem B and two-body quantum system AB, respectively.

The process of the proposed model to calculate the entanglement of a quantum system can be shown in Fig. 1.

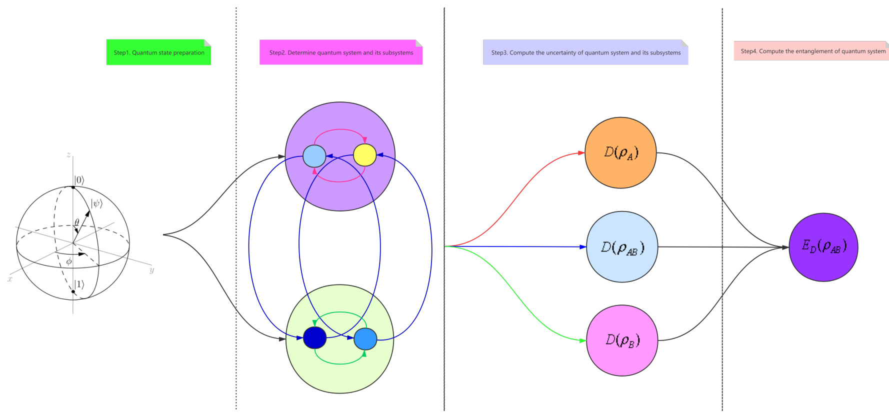

Figure 1: The calculation process of the proposed model

Theorem 3.1. *When the new entropy measure of quantum uncertainty degenerates to classical von Neumann entropy, the quantum entanglement measurement based on belief entropy will degenerate to classical partial entropy entanglement.*

Proof 3.1. *Relying on the equations of Eq.(6), the equation is obtained:*

$$E_D(\rho_{AB}) = \frac{1}{2}\{D(\rho_A) + D(\rho_B) - D(\rho_{AB})\}$$

When the new entropy measure of quantum uncertainty degenerates to classical von Neumann entropy, we can obtain the following equation:

$$\begin{aligned} E_D(\rho_{AB}) &= \frac{1}{2}\{D(\rho_A) + D(\rho_B) - D(\rho_{AB})\} \\ &= \frac{1}{2}\{S(\rho_A) + S(\rho_B) - S(\rho_{AB})\} \\ &= E_l(\rho_{AB}) \end{aligned}$$

In this way, the quantum entanglement measurement based on belief entropy will degenerate to classical partial entropy entanglement. □

4. Numerical examples

Example 4.1. *Suppose the quantum system at this time is a pure state, as shown below:*

$$|\psi >= \frac{1}{\sqrt{2}}|00> + \frac{1}{\sqrt{2}}|11>$$

where,

$$|00>= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, |11>= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The density matrix obtained from the above is as follows:

$$\rho_{AB} = |\psi >< \psi| = \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \end{pmatrix}$$

The reduced density matrix of ρ_{AB} is as follows:

$$\rho_A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \rho_B = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$

The eigenvalues and the corresponding eigenvectors of ρ_{AB} are in following Table 2:

Table 2: The eigenvalues and the corresponding eigenvectors of ρ

Eigenvalues	0	0	0	1
Eigenvectors	$\begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0.7071 \\ 0 \\ 0 \\ -0.7071 \end{pmatrix}$	$\begin{pmatrix} 0.7071 \\ 0 \\ 0 \\ 0.7071 \end{pmatrix}$

Relying on the equation of Eq.(4), the equations are obtained:

$$D(\rho_{AB}) = \ln 3, D(\rho_A) = \ln 2, D(\rho_B) = \ln 2$$

Relying on Eq.(6), we get the entanglement degree of the quantum system ρ_{AB} as follows:

$$E_D(\rho_{AB}) = \frac{1}{2}\{D(\rho_A) + D(\rho_B) - D(\rho_{AB})\} = \frac{1}{2}\{\ln 2 + \ln 2 - \ln 3\}$$

The comparison between the partial entropy entanglement and quantum entanglement measurement based on belief entropy is shown in following Table 3:

Table 3: The comparison of two models

Partial entropy entanglement	The proposed model
$\ln 2$	$\frac{1}{2}\ln\frac{4}{3}$

It can be seen from the above table that the proposed model is different from the partial entropy entanglement, which is related to the problem of von Neumann entropy in measuring the uncertainty of quantum systems.

Example 4.2. *Suppose the quantum system at this time is a pure state, as shown below:*

$$|\psi>=\frac{1}{\sqrt{3}}|00>+\frac{1}{\sqrt{3}}|01>+\frac{1}{\sqrt{3}}|10>$$

where,

$$|00>=\begin{pmatrix}1\\0\\0\\0\end{pmatrix},|01>=\begin{pmatrix}0\\0\\1\\0\end{pmatrix},|10>=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

The density matrix obtained from the above is as follows:

$$\rho_{AB}=|\psi><\psi|=\begin{pmatrix}1/3 & 1/3 & 1/3 & 0\\ 1/3 & 1/3 & 1/3 & 0\\ 1/3 & 1/3 & 1/3 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

The reduced density matrix of ρ_{AB} is as follows:

$$\rho_A=\begin{pmatrix}2/3 & 1/3\\ 1/3 & 1/3\end{pmatrix},\rho_B=\begin{pmatrix}2/3 & 1/3\\ 1/3 & 1/3\end{pmatrix}$$

The eigenvalues and the corresponding eigenvectors of ρ_{AB} are in following Table 4:

Table 4: The eigenvalues and the corresponding eigenvectors of ρ

Eigenvalues	0	0	0	1
Eigenvectors	$\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$	$\begin{pmatrix}-0.8026\\0.5310\\0.2716\\0\end{pmatrix}$	$\begin{pmatrix}-0.1498\\-0.6202\\0.7700\\0\end{pmatrix}$	$\begin{pmatrix}0.5774\\0.5774\\0.5774\\0\end{pmatrix}$

Relying on the equation of Eq.(4), the equations are obtained:

$$D(\rho_{AB})=\ln 7$$

The eigenvalues and the corresponding eigenvectors of ρ_A and ρ_B are in following Table 5:

Table 5: The eigenvalues and the corresponding eigenvectors of ρ_A

Eigenvalues	0.1273	0.8727
Eigenvectors	$\begin{pmatrix}0.5257\\-0.8507\end{pmatrix}$	$\begin{pmatrix}-0.8507\\-0.5257\end{pmatrix}$

Relying on the equation of Eq.(4), the equations are obtained:

$$D(\rho_A) = D(\rho_B) = 1.48$$

Relying on Eq.(6), we get the entanglement degree of the quantum system ρ_{AB} as follows:

$$E_D(\rho_{AB}) = \frac{1}{2}\{D(\rho_A) + D(\rho_B) - D(\rho_{AB})\} = \frac{1}{2}\{1.48 + 1.48 - \ln 7\}$$

The comparison between the partial entropy entanglement and quantum entanglement measurement based on belief entropy is shown in following Table 6:

Table 6: The comparison of two models

Partial entropy entanglement	The proposed model
$\ln 2$	0.93

It can be seen from the above table that the proposed model is different from the partial entropy entanglement, which is related to the problem of von Neumann entropy in measuring the uncertainty of quantum systems.

5. Conclusion

Partial entropy entanglement is a classic tool for measuring the entanglement of a quantum system. Partial entropy entanglement is based on von Neumann entropy, and von Neumann entropy has some problems in measuring the uncertainty of quantum systems, which leads to partial entropy entanglement is not efficient enough to measure the entanglement of quantum systems. This article proposes quantum entanglement measurement based on belief entropy, which is based on the partial entropy entanglement and the new entropy measure of quantum system uncertainty. Because the new entropy measure of quantum system uncertainty is based on the classic von Neumann entropy, and the new entropy measure of quantum system uncertainty is more efficient than von Neumann entropy in measuring the uncertainty of the quantum system. Therefore, quantum entanglement measurement based on belief entropy is more efficient and reasonable in measuring the entanglement of a quantum system than the classical partial entropy entanglement. Numerical examples are used to prove that the quantum entanglement measurement based on belief entropy is more efficient and reliable in measuring the entanglement of quantum systems than the classical partial entropy entanglement. The experimental results show that the quantum entanglement measurement based on belief entropy can measure quantum systems more efficiently and reliably than the classical partial entropy entanglement.

Acknowledgment

The work is partially supported by National Natural Science Foundation of China (Grant No. 61973332), JSPS Invitational Fellowships for Research in Japan (Short-term).

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